Some equivalence relations which are Borel reducible to isomorphism between separable Banach spaces

Valentin Ferenczi* and Elói Medina Galego

Abstract

We prove that the relation E_{K_σ} is Borel reducible to isomorphism and complemented biembeddability between subspaces of c_0 or l_p with $1 \leq p < 2$. We also show that the relation $E_{K_\sigma} \otimes =^+$ is Borel reducible to isomorphism, complemented biembeddability, and Lipschitz isomorphism between subspaces of L_p for $1 \leq p < 2$.

1. Introduction

In this paper, we are mainly interested in the complexity of the relation of isomorphism between separable Banach spaces. The central notion in the theory of classification of analytic equivalence relations on Polish spaces by means of their relative complexity is the concept of Borel reducibility between equivalence relations. This concept originated from the works of H. Friedman and L. Stanley and independently from the works of L. A. Harrington, A. S. Kechris and A. Louveau.[1]

1.1. Borel reducibility of equivalence relations on Polish spaces. Let R (resp. R') be an analytic equivalence relation on a Polish space E (resp. E'). We say that (E, R) is *Borel reducible* to (E', R'), and write $(E, R) \leq_B (E', R')$, if there exists a Borel map f from E to E', such that for all x and y in E,

$$xRy \Leftrightarrow f(x)R'f(y).$$

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One may also restricts one's attention to Borel, instead of analytic, equivalence relations. Note also that the above definition is valid for general relations. A theory of $\leq_B$ for quasi-orders has been recently developed by A. Louveau and C. Rosendal [27].

Observe that if R and R' are equivalence relations, then the induced quotient map from E/R into E'/R' is an injection. In particular, E' has at least as many R'-classes as E has R-classes. In fact, equivalence classes for R' provide invariants for the equivalence relation R, and furthermore this can be obtained in a Borel way. So the order $\leq_B$ can be seen as a measure of relative complexity between analytic equivalence relations on Polish spaces.

The relation (E, R) is *Borel bireducible* to (E', R'), $(E, R) \sim_B (E', R')$, whenever both $(E, R) \leq_B (E', R')$ and $(E', R') \leq_B (E, R)$ hold. Two relations which are Borel bireducible to each other are said to have the same complexity. We write $(E, R) <_B (E', R')$ when $(E, R) \leq_B (E', R')$ but $(E, R) \not\sim_B (E', R')$.

In the theory of classification of analytic equivalence relations on Polish spaces, one tries to classify those relations up to Borel bireducibility. Even for Borel relations, the situation is quite complicated, but there are a number of natural milestones. They correspond to canonical equivalence relations on some classical Polish spaces.

We thus have a scale of canonical relations, and given an equivalence relation on a Polish space, we wish to locate it on this scale of complexity.

1.2. Complexity of analytic equivalence relations on Polish spaces. We give some of these natural milestones and the relations between them. The relation $(n, =)$ of equality on $n \in \mathbb{N}$ is the canonical example of a relation with n classes. We also define $(\omega, =)$ and $(2^\omega, =)$. Because of their cardinalities, it is clear that

$$(1) \qquad (1, =) <_B \ldots <_B (n, =) <_B (\omega, =) <_B (2^\omega, =).$$

The next relation is $(2^\omega, E_0)$, or in short E_0. It is defined on 2^ω by

$$\alpha E_0 \beta \Leftrightarrow \exists m \ \forall n \geq m, \ \alpha(n) = \beta(n),$$

and it is well-known and not difficult to see that it satisfies

$$(2) \qquad (2^\omega, =) <_B E_0.$$

By a theorem of Silver [32] and a theorem of Harrington-Kechris-Louveau [16], this list is extensive for those Borel equivalence relations which are Borel reducible to E_0. This is false for analytic equivalence relations, see [29] for more details.

After E_0, the order is no longer linear and the natural examples fall into one of two groups.

A first family of milestones is given by Borel action of Polish groups on Polish spaces. Given such an action of a Polish group G on a Polish space X, the orbit relation E_G^X on X, defined by $xE_G^Xy \Leftrightarrow \exists g \in G : y = g.x$, is an equivalence relation on X. It is called a *G-equivalence relation* and it is analytic. It turns out that given a Polish group G, there is always a G-equivalence relation which is $\leq_B$-maximum among all possible G-equivalence relations on Polish spaces [2]. This equivalence relation is denoted by E_G^∞, without explicit reference to the Polish space on which it is defined.

Of particular interest are $E_{F_2}^\infty$, where F_2 is the free group with 2 generators, $E_{S_\infty}^\infty$, where S_∞ is the group of permutations of the integers, and $E_{G_0}^\infty$, where G_0 is the group of homeomorphisms of the Hilbert cube.

In fact, $E_{F_2}^\infty$ is $\leq_B$-maximal among Borel equivalence relations on Polish spaces for which each equivalence class is countable or equivalently, among G-equivalence relations for countable groups G [11], [13]. The relation $E_{G_0}^\infty$ is $\leq_B$-maximum among all G-equivalence relations for Polish groups G (Theorem 9.18 in [21] and Theorem 2.3.5. in [2]). We have

$$E_0 <_B E_{F_2}^\infty <_B E_{S_\infty}^\infty <_B E_{G_0}^\infty. \tag{3}$$

On the other hand, not all Borel equivalence relations are Borel reducible to equivalence relations associated to Borel actions of Polish groups. This is the second family of milestones "on the other side".

The relation E_1 [23] is defined on $\mathbb{R}^\omega$ by

$$\alpha E_1 \beta \Leftrightarrow \exists m \ \forall n \geq m,\ \alpha(n) = \beta(n).$$

It is not reducible to any G-equivalence relation for any Polish group G [23].

There exists a $\leq_B$-maximum equivalence relation E_{K_σ} among K_σ equivalence relations [31]. It is said to be *K_σ-complete* and satisfies

$$E_1 <_B E_{K_\sigma}. \tag{4}$$

Rosendal [31] has found useful representations of this equivalence relation, we will use the following one, which was actually the starting point for this paper. Let X_0 be the set $\Pi_{n\geq 1} n$. The relation H_0 on X_0 is defined by

$$\alpha H_0 \beta \Leftrightarrow \exists N \ \forall k, |\alpha(k) - \beta(k)| \leq N.$$

In [31] it was proved that the relation H_0 is K_σ-complete, that is, Borel bireducible with E_{K_σ}.

We may add canonical examples to our list using the operation $+$ defined as follows, see e.g. [12]. Let E be an analytic equivalence relation on a Polish space X. Then E^+ is the (also analytic) equivalence relation defined on X^ω by

$$(x_n)E^+(y_n) \Leftrightarrow \forall n\ \exists m, p : (x_n E y_m) \wedge (y_n E x_p).$$

For example, $(2^\omega, =)^+$, also written $=^+$, is the relation of equality of countable subsets of 2^ω. By properties of $E^\infty_{S_\infty}$ and the 'jump' properties of $+$, see [12], or [18] (where $=^+$ is called $E_{countable}$), it satisfies

$$E^\infty_{F_2} <_B =^+ <_B E^\infty_{S_\infty} \tag{5}$$

Finally, it is also known that there exists a $\leq_B$-maximal element among analytic equivalence relations on Polish space, it is denoted by $E_{\Sigma^1_1}$ [27]. Representations of this relation are for example isometric biembeddability between separable Banach spaces or isometric biembeddability between metric Polish spaces [27].

The $\leq_B$-relations (1), (2), (3), (4) and (5) can be summarized as follows:

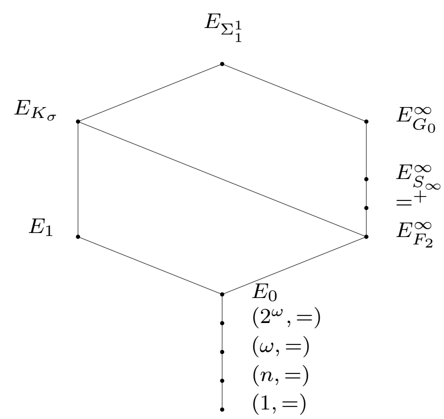

Figure 1: simplified diagram of complexity of analytic equivalence relations on Polish spaces.

1.3. Complexity of isomorphism of separable Banach spaces. Now if one is interested in Banach space theory, there are two possible directions in relation with the theory of classification of analytic equivalence relations up to Borel bireducibility.

The first one is to determinate the general $\leq_B$ complexity of isomorphism between separable Banach spaces, or at least a lower bound for this complexity. In other words, to show that isomorphism between separable Banach spaces reduces rather complex equivalence relations.

The question is also asked for other natural equivalence relations of interest in Banach space theory, such as Lipschitz isomorphism, biembeddability, complemented biembeddability, isometry and so on.

There is a natural setting for this question. All separable Banach spaces can be seen as subspaces of an isometrically universal Banach space such as $C([0,1])$. The set of such subspaces can be equipped with the Effros-Borel structure, see e.g. [3], which turns it into a standard Borel space.

However in practice, one deals with particular examples, like subspaces generated by subsequences of a given basis, or block-subspaces of a given basis, and uses an ad-hoc topology associated to the set of Banach spaces used for the reduction. For example, all Schauder bases can be seen as subsequences of the universal basis of Pełczyński [25], and thus a Banach space with a basis can be represented as an equivalence class on 2^ω.

There are only a few results in that direction, and they are recent. B. Bossard [3] proved that isomorphism between Banach spaces is analytic non-Borel, and Borel reduces E_0. Using Tsirelson's space, Rosendal [29] improved the result to E_1, which implies that isomorphism between separable Banach spaces is not associated to the Borel action of a Polish group on a Polish space.

The other direction of research is to try to relate the complexity of the relation of isomorphism between subspaces of a given separable Banach space X to geometrical properties of X.

Indeed, by the solution to the homogeneous Banach space problem, given by W. T. Gowers [15] and R. A. Komorowski, N. Tomczak-Jaegermann [35], if a Banach space X has only one class of isomorphism of subspaces, then X must be isomorphic to l_2. But it is not known if, for example, there exists a Banach space, other than l_2, with at most 2, or even at most ω classes of isomorphism of subspaces. The following question, asked by G. Godefroy, is then natural: if the complexity of isomorphism between subspace of X is low (in the sense of cardinality or more interestingly in the sense of $\leq_B$), then what geometrical, regularity properties must X satisfy?

It turns out that a first natural threshold for this question is the relation E_0. Indeed, recently the first named author and Rosendal [10] have defined a Banach space X to be *ergodic* if E_0 is Borel reducible to isomorphism between subspaces of X and have obtained various results about ergodic spaces.

Rosendal [30] noticed that hereditarily indecomposable Banach spaces are ergodic, and also proved that an unconditional basis of a non-ergodic Banach space must have a subsequence such that all further subsequences

span isomorphic susbpaces. In [9] and [30] it is proved that a non-ergodic Banach space X with an unconditional basis must be isomorphic to its square, its hyperplanes and more generally to $X \oplus Y$ for any subspace Y generated by a subsequence of the basis. This was already obtained by Kalton [20] for spaces with an unconditional basis and only countably many classes. Finally, in [10] it is proved that a non-ergodic Banach space must contain a subspace X with an unconditional basis which is isomorphic to $X \oplus Y$ for all block-subspaces Y of X.

In this paper, we work with the classical Banach spaces c_0, l_p and L_p, $1 \leq p < 2$, to provide new results concerning the two directions of research.

1.4. Organization of the paper. In the next section we Borel reduce the relation E_{K_σ} to isomorphism between subspaces of $l_p, 1 \leq p < 2$ (Theorem 2.6). Our main tool for this is a theorem of P. G. Casazza and N. Kalton about uniqueness of unconditional structure for Banach spaces, see Theorem 2.3. This result will allow us to prove that certain spaces with unconditional bases are isomorphic if and only if their canonical bases are equivalent. It was Kalton who suggested that this paper contained an answer to the problem of the number of non-isomorphic subspaces of l_p.

In the third section we also reduce the relation E_{K_σ} to isomorphism between subspaces of c_0 (Theorem 3.3). There we use another theorem of P. G. Casazza and N. Kalton (Theorem 3.1) which is a strengthening of Theorem 2.3 in the case of c_0-sums. In particular, our results show that c_0 and $l_p, 1 \leq p < 2$ are ergodic. This extends to Banach spaces with an unconditional basis with the shift property and satisfying a lower p-estimate, $1 \leq p < 2$ (Theorem 2.7).

In the fourth section, using more simple techniques, we reduce the relation $=^+$ to isomorphism between subspaces of L_p, $1 \leq p < 2$ (Theorem 4.1). In combination with Theorem 2.6, we deduce that $E_{K_\sigma} \otimes =^+$ is Borel reducible to isomorphism between subspaces of $L_p, 1 \leq p < 2$. Thus isomorphism between separable Banach spaces is not reducible to the equivalence relation associated to the Borel action of a Polish group, but reduces G-equivalence relations for non-trivial actions of such groups G.

Finally, in the last section we present a diagram (figure 2) containing known facts about complexity of isomorphism between subspaces of a Banach space. Then, we point out some open problems and a conjecture related to our results.

1.5. Notation. We shall write $X \simeq Y$ to mean that two Banach spaces X and Y are isomorphic, $X \overset{L}{\simeq} Y$ to mean that they are Lipschitz isomorphic, $X \overset{c}{\hookrightarrow} Y$ to mean that X is isomorphic to a complemented subspace of Y, and $X \overset{c}{\simeq} Y$ to mean that $X \overset{c}{\hookrightarrow} Y$ and $Y \overset{c}{\hookrightarrow} X$. For Banach spaces X and Y written as c_0-sums or l_p-sums of l_q^n spaces, we shall abuse notation by writing $X \sim Y$ to mean that the canonical bases are equivalent.

If $(X_n)_{n\in\mathbb{N}}$ is a sequence of Banach spaces, we will denote the l_p-sum $(\sum_{n\in\mathbb{N}} \oplus X_n)_{l_p}$ of the X_n's by $l_p(X_n)_{n\in\mathbb{N}}$. The l_p-sum of infinitely many copies of a Banach space X is denoted as usual $l_p(X)$. If the Banach spaces X_n, $n \in \mathbb{N}$, are given with canonical bases, then $l_p(X_n)_{n\in\mathbb{N}}$ has a corresponding canonical basis associated to a given bijection between $\mathbb{N}$ and $\mathbb{N}^2$. We use the similar notation for c_0-sums.

2. Reductions of E_{K_σ} to isomorphism between subspaces of l_p, $1 \le p < 2$.

Rosendal [30] proved that E_{K_σ} is Borel reducible to equivalence between Schauder basis in Banach spaces. We start by rephrasing his proof in a slightly higher generality (Lemma 2.1 and Corollary 2.2).

Lemma 2.1: *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of integers, $(p_n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$ be bounded sequences of reals larger than 1 and $1 \le p < +\infty$. Then*

$$l_p(l_{p_n}^{K_n})_{n\in\mathbb{N}} \sim l_p(l_{q_n}^{K_n})_{n\in\mathbb{N}} \Leftrightarrow \exists C > 0, \forall n \in \mathbb{N}, |p_n - q_n| \le \frac{C}{\log K_n},$$

and the similar result is valid for c_0-sums.

Proof. By a classical consequence of Hölder's inequality, the constant of equivalence c_n between the canonical bases of $l_{p_n}^{K_n}$ and $l_{q_n}^{K_n}$ is $K_n^{|1/p_n - 1/q_n|}$ [34].

Let c be an upper bound for the sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$, then

$$e^{|p_n - q_n| \log K_n / c^2} \le c_n \le e^{|p_n - q_n| \log K_n}.$$

It follows that if the canonical bases of $l_p(l_{p_n}^{K_n})_{n\in\mathbb{N}}$ and $l_p(l_{q_n}^{K_n})_{n\in\mathbb{N}}$ (resp. $c_0(l_{p_n}^{K_n})_{n\in\mathbb{N}}$ and $c_0(l_{q_n}^{K_n})_{n\in\mathbb{N}}$) are C-equivalent, then for all n, $|p_n - q_n| \le c^2 \log C / \log K_n$.

Conversely, if for all n, $|p_n - q_n| \le M/\log K_n$, then the canonical bases of $l_p(l_{p_n}^{K_n})_{n\in\mathbb{N}}$ and $l_p(l_{q_n}^{K_n})_{n\in\mathbb{N}}$ (resp. $c_0(l_{p_n}^{K_n})_{n\in\mathbb{N}}$ and $c_0(l_{q_n}^{K_n})_{n\in\mathbb{N}}$) are $(e^M)^2$-equivalent. ∎

Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of integers, $(p_n)_{n\in\mathbb{N}}$ be a sequence of real numbers greater than 1 and $1 \le p < +\infty$. We recall that X_0 denotes the set

$\Pi_{n\geq 1}n$, and that the relation H_0 on X_0 is defined by

$$\alpha H_0 \beta \Leftrightarrow \exists N \ \forall k, |\alpha(k) - \beta(k)| \leq N.$$

For $\alpha \in X_0$, we denote by $l_p(l_{p_n}^{K_n}(\alpha))$ the Banach space

$$l_p(l_{p_n}^{K_n}(\alpha)) = (\sum_n \oplus l^{K_n}_{p_n + \frac{\alpha(n)}{\log K_n}})_{l_p},$$

and we use the similar definition for c_0-sums.

Corollary 2.2: *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of integers, $(p_n)_{n\in\mathbb{N}}$ be a sequence of reals larger than 1 such that $(p_n + \frac{n}{\log K_n})_{n\in\mathbb{N}}$ is bounded and $1 \leq p < +\infty$. Then for all α and β in X_0,*

$$\alpha H_0 \beta \Leftrightarrow l_p(l_{p_n}^{K_n}(\alpha)) \sim l_p(l_{p_n}^{K_n}(\beta)),$$

and the similar result is valid for c_0-sums.

It is known that for $1 \leq p \leq r \leq 2$ and $\epsilon > 0$, l_p contains $1+\epsilon$-isomorphic copies of l_r^n, in fact L_r is isometric to some subspace of L_p, see e.g. [25]. It follows that for any sequence $(p_n)_{n\in\mathbb{N}}$ of reals such that for all n, $p \leq p_n \leq 2$ and any sequence of integers $(K_n)_{n\in\mathbb{N}}$, the space $l_p(l_{p_n}^{K_n})_{n\in\mathbb{N}}$ is isomorphic to a subspace of l_p.

Our main ingredient will be the following theorem of Casazza and Kalton [6], which can be thought of as a first step towards uniqueness of unconditional structure for Banach spaces which are sufficiently far from l_2.

We refer to [25], [19] for the definition of and background about Banach lattices. If X and Y are Banach lattices, a bounded linear operator $V : X \to Y$ is called a *lattice homomorphism* if $V(x_1 \vee x_2) = Vx_1 \vee Vx_2$ for all $x_1, x_2 \in X$. Following [6], define a Banach lattice X to be *sufficiently lattice-euclidean* if there exists $C \geq 1$ such thar for all $n \in \mathbb{N}$, there exist operators $S : X \to l_2^n$ and $T : l_2^n \to X$ such that $ST = I_{l_2^n}$, $\|S\|\|T\| \leq C$ and such that S is a lattice homomorphism. This is equivalent to saying that l_2 is finitely representable as a complemented sublattice of X.

A Banach space with a 1-unconditional basis $(x_n)_{n\in\mathbb{N}}$ is naturally considered as a Banach lattice by defining

$$\sum_{n\in\mathbb{N}} a_n x_n \geq 0 \Leftrightarrow \forall n \in \mathbb{N}, a_n \geq 0.$$

It is classical to consider a Banach space X with a C-unconditional basis, $C \geq 1$, as a Banach lattice as well, with the same definition of $\leq$ and with the

restriction of having to add a constant in some inequalities; alternatively, one may equip X with an equivalent norm which turns $(x_n)_{n\in\mathbb{N}}$ 1-unconditional, and the results concerning the Banach lattice structure of X can be transfered back to the initial norm, up to to the constant of equivalence.

For an unconditional basis $(x_n)_{n\in\mathbb{N}}$ of a Banach space (seen as a Banach lattice), being sufficiently lattice-euclidean is the same as having, for some $C \geq 1$ and every $n \in \mathbb{N}$, a C-complemented, C-isomorphic copy of l_2^n whose basis is disjointly supported on $(x_n)_{n\in\mathbb{N}}$.

Theorem 2.3: (Casazza -Kalton [6]) *Let X be a Banach space with an unconditional basis and $(y_n)_{n\in\mathbb{N}}$ be an unconditional basic sequence in X which is not sufficiently lattice-euclidean and spans a complemented subspace of X. Then $(y_n)_{n\in\mathbb{N}}$ is equivalent to a sequence of disjointly supported vectors which spans a complemented subspace in X^N for some N.*

Let X be a Banach space with a Schauder decomposition $X = \sum_{i=1}^{+\infty} \oplus X_i$. We shall say that vectors x and y in X are *successive* and write $x < y$ if there exists intervals of integers E and F such that $\max(E) < \min(F)$, $x \in \sum_{i\in E} X_i$, and $y \in \sum_{i\in F} X_i$.

We say that the Schauder decomposition of X satisfies a *lower p-estimate with constant* $C \geq 1$ if for any successive vectors $x_1 < \cdots < x_n$ in X, $(\sum_{i=1}^n \|x_i\|^p)^{1/p} \leq C\|\sum_{i=1}^n x_i\|$. For $1 \leq p \leq +\infty$, the conjugate p' of p is as usual defined by $\frac{1}{p} + \frac{1}{p'} = 1$ (with $\frac{1}{+\infty} = 0$).

Lemma 2.4: *Fix $1 \leq p < 2$, $(K_n)_{n\in\mathbb{N}}$ a sequence of integers and $(p_n)_{n\in\mathbb{N}}$ a sequence of real numbers which is bounded below by p. Let $r = \sup_{n\in\mathbb{N}} p_n$ and r' be the conjugate of r. Suppose $X = \sum_{n=1}^{+\infty} \oplus l_{p_n}^{K_n}$ is a Schauder decomposition of X satisfying a lower p-estimate with constant $C \geq 1$. Then for all $k \in \mathbb{N}$, for all vectors $y_1, \ldots, y_k$ in X which are disjointly supported on its canonical basis,*

$$\sum_{i=1}^k \|y_i\| \leq C k^{1/r'} \|\textstyle\sum_{i=1}^k y_i\|.$$

Proof. We may assume that $r < +\infty$. Let $y_1, \ldots, y_k$ be as above. For each $1 \leq i \leq k$, we write $y_i = \sum_{n=1}^{+\infty} y_{in}$, where y_{in} is the projection of y_i onto the $l_{p_n}^{K_n}$ summand. Then

$$C\|\textstyle\sum_{i=1}^k y_i\| = C\|\sum_{n=1}^{+\infty}\sum_{i=1}^k y_{in}\| \geq (\displaystyle\sum_{n=1}^{+\infty}(\sum_{i=1}^k \|y_{in}\|^{p_n})^{\frac{p}{p_n}})^{\frac{1}{p}}.$$

Denote $a_{in} = \|y_{in}\|^p$ and $\alpha = r/p$, $\alpha_n = p_n/p$. Then

$$C^p \| \sum_{i=1}^k y_i \|^p \geq \sum_{n=1}^{+\infty} (\sum_{i=1}^k a_{in}^{\alpha_n})^{\frac{1}{\alpha_n}} \geq \sum_{n=1}^{+\infty} k^{-1/\alpha_n{}'} \sum_{i=1}^k a_{in},$$

by Hölder's inequality, since $\alpha_n \geq 1$. Now for every $n \in \mathbb{N}$, $\alpha \geq \alpha_n$, so

$$C^p \| \sum_{i=1}^k y_i \|^p \geq k^{-1/\alpha'} \sum_{n=1}^{+\infty} \sum_{i=1}^k a_{in}.$$

On the other hand,

$$(\sum_{i=1}^k \|y_i\|)^p = (\sum_{i=1}^k (\sum_{n=1}^{+\infty} a_{in})^{\frac{1}{p}})^p \leq k^{\frac{p}{p'}} \sum_{i=1}^k (\sum_{n=1}^{+\infty} a_{in}),$$

once again by Hölder's inequality, since $p \geq 1$. Finally,

$$(\sum_{i=1}^k \|y_i\|)^p \leq C^p k^{\frac{p}{p'}+\frac{1}{\alpha'}} \| \sum_{i=1}^k y_i \|^p,$$

so

$$\sum_{i=1}^k \|y_i\| \leq C k^{\frac{1}{p'}+\frac{1}{p\alpha'}} \| \sum_{i=1}^k y_i \|,$$

and the fact that $\frac{1}{p'} + \frac{1}{p\alpha'} = \frac{1}{r'}$ concludes the proof. ∎

To prove the next proposition we need to recall that two unconditional sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ in a Banach space X are said to be *permutatively equivalent* if there is a permutation π of $\mathbb{N}$ so that $(u_n)_{n\in\mathbb{N}}$ and $(v_{\pi(n)})_{n\in\mathbb{N}}$ are equivalent.

Proposition 2.5: *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of integers, $(p_n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$ sequences of real numbers and $1 \leq p < 2$. Assume*
(1) *$p < p_n < 2$ and $p < q_n < 2$, for all $n \in \mathbb{N}$;*
(2) *$(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ are decreasing sequences:*
(3) *$K_1 \geq 4$ and $K_n \geq n^2 K_{n-1}$, for all $n \geq 2$.*

Then whenever $l_p(l_{p_n}^{K_n})_{n\in\mathbb{N}} \overset{c}{\hookrightarrow} l_p(l_{q_n}^{K_n})_{n\in\mathbb{N}} \oplus F$, for some finite-dimensional space F, there exists $C > 0$ such that $p_n - q_n \leq C/\log K_n$, for all $n \in \mathbb{N}$.

Proof. Note that by Lemma 2.4, any disjointly supported sequence of vectors $x_1, \ldots, x_k$ in $l_p(l_{p_n}^{K_n})_{n\in\mathbb{N}}$ satisfies

$$\sum_{i=1}^k \|x_i\| \leq k^{1/q_1'} \| \sum_{i=1}^k x_i \|,$$

and $q_1{}' > 2$. So $l_p(l_{p_n}^{K_n})_{n\in\mathbb{N}}$ is not sufficiently lattice-euclidean. By Theorem 2.3, for some N, the canonical basis of $l_p(l_{p_n}^{K_n})_{n\in\mathbb{N}}$ is C-equivalent to a disjointly supported sequence in $(l_p(l_{q_n}^{K_n})_{n\in\mathbb{N}} \oplus F)^N$. Modifying N and C we may assume $F = \{0\}$. Then without loss of generality we may write this space as $l_p(l_{q_n}^{NK_n})_{n\in\mathbb{N}}$ (the canonical bases are permutatively equivalent). Take

$k \geq k(N)$, where $k(N)$ is such that this condition ensures $\frac{K_k}{2} \geq \sum_{i=1}^{k-1} NK_i$; it exists by condition (3). The canonical basis of $l_{p_k}^{K_k}$ is C-equivalent to a disjointly supported sequence in $l_p(l_{q_n}^{NK_n})_{n\in\mathbb{N}}$. By the condition on K_k, we see that the canonical basis $(e_i)_{i\in\mathbb{N}}$ of $l_{p_k}^{K_k/2}$ is C-equivalent to a disjointly supported sequence $(f_i)_{i\in\mathbb{N}}$ in $(\sum_{n\geq k} \oplus l_{q_n}^{NK_n})_{l_p}$. We may now apply Lemma 2.4 to the sequence $(f_i)_{i\in\mathbb{N}}$. As $q_k = \max\{q_n, n \geq k\}$,

$$C(K_k/2)^{1/p_k} \geq \| \textstyle\sum_{i=1}^{K_k/2} f_i\| \geq (K_k/2)^{-1/q_k'} \displaystyle\sum_{i=1}^{K_k/2} \|f_i\| \geq (K_k/2)^{1/q_k}/C.$$

Consequently

$$(K_k/2)^{1/q_k - 1/p_k} \leq C^2,$$

and

$$p_k - q_k \leq 4(1/q_k - 1/p_k) \leq 8 \log C / \log(K_k/2) \leq 16 \log C / \log K_k.$$

This is true for any $k \geq k(N)$, so the proposition is proved. ■

Theorem 2.6: *Suppose $1 \leq p < 2$. Then the relation E_{K_σ} is Borel reducible to isomorphism, to Lipschitz isomorphism and to complemented biembeddability between subspaces of l_p. Indeed, there exist a sequence of integers $(K_n)_{n\in\mathbb{N}}$, a sequence of reals $(p_n)_{n\in\mathbb{N}}$ with $p < p_n < 2$ for all n, such that the following are equivalent for all α and β in X_0:*

(1) $\alpha H_0 \beta$.

(2) $l_p(l_{p_n}^{K_n}(\alpha)) \sim l_p(l_{p_n}^{K_n}(\beta))$.

(3) $l_p(l_{p_n}^{K_n}(\alpha)) \simeq l_p(l_{p_n}^{K_n}(\beta))$.

(4) $l_p(l_{p_n}^{K_n}(\alpha)) \overset{L}{\simeq} l_p(l_{p_n}^{K_n}(\beta))$.

(5) $l_p(l_{p_n}^{K_n}(\alpha)) \overset{c}{\simeq} l_p(l_{p_n}^{K_n}(\beta))$.

Proof. We choose $(K_n)_{n\in\mathbb{N}}$ satisfying (3) of Proposition 2.5, and $(p_n)_{n\in\mathbb{N}}$ such that $p_1 + 1/\log K_1 < 2$, $p < p_n < 2$ for all n, and $\frac{n+1}{\log K_{n+1}} \leq p_n - p_{n+1}$. This is certainly possible if $\sum_{n+1}^{+\infty} \frac{n}{\log K_n}$ is small enough. Then it is clear that the conditions of Proposition 2.5 are satisfied for any two sequences $(p_n + \frac{\alpha(n)}{\log K_n})_{n\in\mathbb{N}}$ and $(p_n + \frac{\beta(n)}{\log K_n})_{n\in\mathbb{N}}$. It follows that (5) implies (1). That (4) implies (5), that is, Lispchitz isomorphism implies complemented biembeddability, comes from the fact that the spaces considered are separable dual spaces (Theorem 2.4 in [17]). (1) implies (2) by Lemma 2.1 and Corollary 2.2 and the rest is obvious. ■

Using a similar proof as in the previous theorem we get the following result. An unconditional basis for a Banach space X is said to have the

shift property if any normalized block-sequence $(x_n)_{n\in\mathbb{N}}$ in X is equivalent to $(x_{n+1})_{n\in\mathbb{N}}$.

Theorem 2.7: *Let X be a Banach space with an unconditional basis with the shift property, which satisfies a lower p-estimate for some $1 \leq p < 2$. Then X is ergodic.*

Proof. Let X be such a space. By Krivine's theorem (see e.g. [28]), l_r is block-finitely represented in X for some r, and by the lower estimate, we have that $r \leq p$. But then all $l_{r'}$ for $r \leq r' \leq 2$, and in particular $p \leq r' \leq 2$ are finitely represented in X (with constant 2 say). We may then associate to each $\alpha \in X_0$ a subspace $X(\alpha)$ of X which is a direct sum on the basis of $l_{p_n}^{K_n}$s for some p_ns in $]p, 2[$ as previously. The canonical Schauder decomposition of the space $X(\alpha)$ satisfies a lower p-estimate and is unconditional. Also each $l_{p_n}^{K_n}$ has a canonical 1-unconditional basis and X satisfies the shift property, so by [5], Proposition 2.3, the canonical basis of $X(\alpha)$ is unconditional. We may thus follow the proof of Proposition 2.5 (note that if $X = \sum \oplus l_{q_n}^{K_n}$ satisfies a lower p-estimate with constant C, then $X^N \simeq \sum \oplus l_{q_n}^{NK_n}$ satisfies a lower p-estimate as well, with a constant depending on C, N and p). We get that

$$X(\alpha) \simeq X(\beta) \Rightarrow \alpha H_0 \beta.$$

It doesn't seem easy to get the converse without more information on the norm on X. We shall in fact reduce E_0 instead of E_{K_σ}. For this, consider 2^ω as a subset of X_0 by $j((\alpha(n))_{n\in\mathbb{N}}) = (0, \alpha(1), 2\alpha(2), 3\alpha(3), \ldots)$. Then clearly, for any $\alpha,\ \beta \in 2^\omega$, $\alpha E_0 \beta$ if and only if $j(\alpha) H_0 j(\beta)$, so from the above,

$$X(j(\alpha)) \simeq X(j(\beta)) \Rightarrow \alpha E_0 \beta.$$

But we also have

$$\alpha E_0 \beta \Rightarrow X(j(\alpha)) \simeq X(j(\beta)),$$

because if $\alpha E_0 \beta$ then $X(j(\alpha))$ and $X(j(\beta))$ have canonical bases which differ by only a finite number of vectors. So E_0 is Borel reducible to isomorphism between subspaces of X. ∎

3. Reductions of E_{K_σ} to isomorphism between subspaces of c_0.

We now turn our attention to spaces of the form $c_0(l_{p_n}^{K_n}(\alpha))$. Note that as a c_0-sum of finite-dimensional spaces, every such space is isomorphic to a subspace of c_0. The previous results concerning isomorphism and complemented biembeddability between subspaces of l_p, $1 \leq p < 2$ extend by duality to quotients of $l_p, p > 2$ and c_0, and thus by a classical theorem (Theorem 2.f.6

in [25]), also to subspaces of c_0. However, we shall improve these results by also reducing E_{K_σ} to complemented embeddability between subspaces of c_0.

We recall that the definition of $\leq_B$ still makes sense when the relation is not an equivalence relation. In particular, the $\leq_B$-classification of quasi-orders has consequences in the $\leq_B$-classification of equivalence relations, see [27].

Theorem 3.1: (Casazza-Kalton [7]) *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of integers and $(q_n)_{n\in\mathbb{N}}$ a decreasing sequence of reals converging to 1. Then any unconditional basis of a complemented subspace of $c_0(l_{q_n}^{K_n})_{n\in\mathbb{N}}$ is permutatively equivalent to the canonical basis of $c_0(l_{q_n}^{M_n})_{n\in\mathbb{N}}$, for some sequence $(M_n)_{n\in\mathbb{N}}$ such that M_n/K_n is bounded.*

Since it is only implicit in their paper, we sketch how this theorem follows from their results. We also refer to their paper for some definitions which we would not use afterwards.

Proof of Theorem 3.1. Let $(u_k)_{k\in\mathbb{N}}$ be an unconditional basis of a complemented subspace of $c_0(l_{q_n}^{K_n})_{n\in\mathbb{N}}$. According to [7] Corollary 2.5 and [7] Theorem 1.1 we may assume that the (u_k)'s are disjointly supported. By [7] Theorem 3.2 we may assume that for all n and k, $\|u_k(n)\|_{l_{q_n}^{K_n}} = 0$ or 1, and that there exists a partition $\mathbb{N} = \cup_{n\in A} B_n$ of $\mathbb{N}$, such that the space spanned by $(u_k)_{k\in\mathbb{N}}$ is a c_0-sum of the spaces spanned by $(u_k)_{k\in B_n}$, and a C such that for each n, $(u_k)_{k\in B_n}$ is C-complemented, C-tempered (see the definition in [7]). By the Claim in [7] Theorem 3.4, we see that for some K, each $(u_k)_{k\in B_n}$ must be K permutatively equivalent to the canonical basis of $(\sum_{k\in D_n} \oplus l_{q_k}^{P_k})_{c_0}$, with $|D_n| \leq K$ and $P_k/K_k \leq K$. Furthermore, by (3) of [7] Theorem 3.2, for any k, the number of n's such that $k \in D_n$ is uniformly bounded. The theorem follows. ∎

Proposition 3.2: *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of integers and $(p_n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$ be sequences of reals. Assume*

(1) *The sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ are decreasing to 1;*

(2) $1 < p_n + \frac{n}{\log K_n} < 2$ *and* $1 < q_n + \frac{n}{\log K_n} < 2$*, for all $n \in \mathbb{N}$;*

(3) *The sequence $(\frac{n}{\log K_n})_{n\in\mathbb{N}}$ is decreasing;*

(4) $|q_n - p_m| \geq \min(m,n)/\log K_{\min(m,n)}$*, for all $m \neq n$.*

Then whenever $c_0(l_{p_n}^{K_n})_{n\in\mathbb{N}} \overset{c}{\hookrightarrow} c_0(l_{q_n}^{K_n})_{n\in\mathbb{N}}$, it follows that $c_0(l_{p_n}^{K_n})_{n\in\mathbb{N}} \sim c_0(l_{q_n}^{K_n})_{n\in\mathbb{N}}$.

Proof. By Theorem 3.1, the canonical basis of $c_0(l_{p_n}^{K_n})_{n\in\mathbb{N}}$, being equivalent

to an unconditional basis of a complemented subspace of $c_0(l_{q_n}^{K_n})_{n\in\mathbb{N}}$, must be C-permutatively equivalent to the canonical basis of a space $c_0(l_{q_n}^{M_n})_{n\in\mathbb{N}}$, for some constant C and some sequence $(M_n)_{n\in\mathbb{N}}$ of integers.

Let us now fix $n \in \mathbb{N}$. Thus by the above, and using the symmetry of the canonical bases of spaces l_p, there exists for $i = 1, \ldots, k$ integers A_i, such that $K_n = \sum_{i=1}^{k} A_i$, and an increasing sequence of integers $n_i, i \leq k$, such that the canonical basis of $l_{p_n}^{K_n}$ is C-equivalent to the canonical basis of $(\sum_{1\leq i\leq k} \oplus l_{q_{n_i}}^{A_i})_{c_0}$.

We first note that in particular the canonical basis of l_∞^k is C-equivalent to the canonical basis of $l_{p_n}^k$, from which it follows that

$$k^{1/p_n} \leq C,$$

therefore $k \leq C^{p_n} \leq C^2$.

It follows that $\max_i A_i \geq K_n/C^2$. Let i be an integer where this maximum is attained and let $N = n_i$. From the fact that the canonical bases of $l_{p_n}^{A_i}$ and $l_{q_N}^{A_i}$ are C-equivalent, it follows that

$$A_i^{|\frac{1}{p_n}-\frac{1}{q_N}|} \leq C,$$

so by condition (2),

$$1/4|p_n - q_N| \log A_i \leq \log C.$$

We now assume that $n > 8 \log C$, it follows from (2) that $\log K_n > 4 \log C$, and therefore

$$|p_n - q_N| \leq 4 \log C/(\log K_n - 2 \log C) \leq 8 \log C/\log K_n.$$

Now if $N \neq n$, then by conditions (3) and (4),

$$|p_n - q_N| \geq \min(n, N)/\log K_{\min(n,N)} \geq n/\log K_n.$$

But this contradicts the assumption that $n > 8 \log C$. It follows that $N = n$. In particular in the previous inequality, we get

$$|p_n - q_n| \leq 8 \log C/\log K_n.$$

Finally, for all $n > 8 \log C$, $|p_n - q_n| \leq 8 \log C/\log K_n$, and by Corollary 2.2, this means that $c_0(l_{p_n}^{K_n})_{n\in\mathbb{N}} \sim c_0(l_{q_n}^{K_n})_{n\in\mathbb{N}}$. ■

Theorem 3.3: *The relation E_{K_σ} is Borel reducible to isomorphism and to complemented embeddability between subspaces of c_0. Indeed there exist a sequence of integers $(K_n)_{n\in\mathbb{N}}$ and a sequence of reals $(p_n)_{n\in\mathbb{N}}$ such that for α and β in X_0, the following statements are equivalent:*

(1) $\alpha H_0 \beta$.

(2) $c_0(l_{p_n}^{K_n}(\alpha)) \sim c_0(l_{p_n}^{K_n}(\beta))$.

(3) $c_0(l_{p_n}^{K_n}(\alpha)) \simeq c_0(l_{p_n}^{K_n}(\beta))$.
(4) $c_0(l_{p_n}^{K_n}(\alpha)) \overset{c}{\hookrightarrow} c_0(l_{p_n}^{K_n}(\beta))$.

Proof. We choose $(K_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ such that $p_1 + 1/\log K_1 < 2$, p_n is decreasing to 1, $n/\log K_n$ is decreasing, and for all n, $p_n - p_{n+1} \geq 2n/\log K_n$. This is possible if $\sum_{n=1}^{+\infty} \frac{n}{\log K_n}$ is small enough. Then conditions (1), (2), (3) and (4) of Proposition 3.2 are achieved for any two sequences $(p_n + \frac{\alpha(n)}{\log K_n})_{n\in\mathbb{N}}$ and $(p_n + \frac{\beta(n)}{\log K_n})_{n\in\mathbb{N}}$. Corollary 2.2 gives that (1) implies (2). Finally, (4) implies (1) comes from Proposition 3.2. ∎

Remark 3.4: Observe that we cannot use Pełczyński's decomposition method here to show that isomorphism and complemented bi-embeddability coincide, because the conditions we need to impose on $(p_n)_{n\in\mathbb{N}}$ and $(K_n)_{n\in\mathbb{N}}$ prevent the sequence $((p_n - p_{n+1})\log K_n)_{n\in\mathbb{N}}$ from being bounded; that condition is needed to prove that a c_0-sum of $l_{p_n}^{K_n}$'s is isomorphic to its square.

We only used Banach spaces with unconditional bases. The crucial point in our method is that the spaces considered are isomorphic if and only if their canonical bases are equivalent. As Rosendal proved that equivalence of Schauder bases is K_σ-complete [30], we cannot hope to go further up in the hierarchy of complexity than K_σ with this method. So we now turn to a situation where isomorphism corresponds to permutative equivalence of the canonical bases.

4. Reducing G-equivalence relations to isomorphism between separable Banach spaces

It is well-known that if $(Y_i)_{i\in\mathbb{N}}$ is a sequence of Banach spaces, and if a Banach space X is isomorphic to a subspace of $l_p(Y_i)_{i\in\mathbb{N}}$ for some $p \in [1, +\infty)$, then X is isomorphic to a subspace of $\sum_{i=1}^{n} \oplus Y_i$ for some $n \in \mathbb{N}$ or l_p is isomorphic to a subspace of X. In particular, if for some $p \in [1, +\infty)$, the space l_p is isomorphic to a subspace of $l_{p_0}(l_{p_n})_{n\in\mathbb{N}}$, where $p_n \in [1, +\infty)$ for all $n \in \mathbb{N} \cup \{0\}$, then there exists $n \in \mathbb{N} \cup \{0\}$ such that $p = p_n$. For a proof of these facts, see for example [4] Theorem 1.1.

If $(X_n)_{n\in\mathbb{N}}$ is a sequence of Banach spaces, we shall define $l_p^\infty(X_n)_{n\in\mathbb{N}}$ as an l_p-sum where each X_n appears in infinitely many summands. In other words, $l_p^\infty(X_n)_{n\in\mathbb{N}} \simeq l_p(l_p(X_n))_{n\in\mathbb{N}}$, where for each $n \in \mathbb{N}$, $l_p(X_n)$ denotes the l_p-sum of infinitely many copies of X_n, with permutative equivalence of the canonical bases.

Theorem 4.1: *The relation $=^+$ is Borel reducible to isomorphism, to Lipschitz isomorphism, to biembeddability and to complemented biembeddability between subspaces of L_p, $1 \leq p < 2$.*

Proof. Fixing $1 \leq p < 2$, we let P be a perfect subset of the interval $]p, 2[$. We define for $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in P^\omega$, the Banach space

$$X(\alpha) = l_p^\infty(l_{\alpha_n})_{n\in\mathbb{N}}.$$

This defines a Borel map and we show that it reduces $=^+$ on P^ω to isomorphism between subspaces of L_p. Indeed, first note that $X(\alpha)$ is isomorphic to a subspace of L_p as an l_p-sum of subspaces of L_p. Now if $\alpha =^+ \beta$, then every summand in the l_p-sum $X(\alpha)$ (resp. $X(\beta)$) is a summand in $X(\beta)$ (resp. $X(\alpha)$), and both appear infinitely many times as summands. So $X(\alpha)$ is isometric to $X(\beta)$ (in fact its canonical basis is permutatively equivalent to the canonical basis of $X(\beta)$).

Conversely, assume $X(\alpha)$ embeds in $X(\beta)$. Let $n \in \mathbb{N}$, we see that l_{α_n} is isomorphic to some subspace of the l_p-sum $X(\beta)$. As $\alpha_n \neq p$, it follows that there exists m, such that $\alpha_n = \beta_m$ for some m. Assuming $X(\beta)$ embeds in $X(\alpha)$, we get that $\beta_n = \alpha_q$ for some q. As n was arbitrary, we conclude that $\alpha =^+ \beta$.

Finally we conclude that for α and β in P^ω, $\alpha =^+ \beta$ if and only if $X(\alpha)$ is isometric to $X(\beta)$, resp. isomorphic to, Lipschitz isomorphic to, complementably beimbeddable in, beimbeddable in $X(\beta)$. Once again we used [17] Theorem 2.4, together with reflexivity, to see that Lipschitz equivalence implies complemented biembeddability. ∎

Before the next result we recall that an operator T from a Banach space X into a Banach space Y is strictly singular if there exists no infinite dimensional subspace Z of X such that the restriction of T to Z is an isomorphism onto the image. Two Banach spaces X and Y are said to be totally incomparable if X and Y have no isomorphic closed subspaces of infinite dimension.

Theorem 4.2: (Wojtasczyk [36]) *Assume that X_1 and X_2 are Banach spaces such that any operator from X_1 to X_2 is strictly singular. Let X be a complemented subspace of $X_1 \oplus X_2$. Then X is isomorphic to $Y_1 \oplus Y_2$, where Y_i is a complemented subspace of X_i for $i = 1, 2$.*

Given R (resp. R') an equivalence relation on a set E (resp. E'), the product $R \otimes R'$ is defined on $E \times E'$ by

$$(x, x')\ R \otimes R'\ (y, y') \Leftrightarrow xRx' \wedge yRy'.$$

Theorem 4.3: *The relation $E_{K_\sigma} \otimes =^+$ is Borel reducible to isomorphism, Lipschitz isomorphism and complemented biembeddability between subspaces of L_p, $1 \leq p < 2$.*

Proof. Fix $1 \leq p < 2$. Let f be a map given by Theorem 2.6 which Borel reduces E_{K_σ} to isomorphism between subspaces of $l_{(p+1)/2}$. Let P be a perfect subset of $](p+1)/2, 2[$ and g be a map given by Theorem 4.1 which Borel reduces $=^+$ on P^ω to isomorphism between l_p-sums of l_{q_n}-spaces for sequences (q_n) in P. By the result at the beginning of this section, $l_{(p+1)/2}$ is totally incomparable with such l_p-sums. Using Theorem 4.2, we check that the direct sum h of the two maps (defined by $h(\alpha, \beta) = f(\alpha) \oplus g(\beta)$) Borel reduces $E_{K_\sigma} \otimes =^+$ to isomorphism between subspaces of L_p.

Indeed, first note that by construction $h(\alpha, \beta)$ is a subspace of L_p, for α in X_0 and β in P^ω. Then assume α and α' in X_0 are H_0-related, and β and β' in P^ω satisfy $\beta =^+ \beta'$; then $h(\alpha, \beta) \simeq h(\alpha', \beta')$. Conversely, assume $h(\alpha, \beta) \overset{c}{\simeq} h(\alpha', \beta')$. Then in particular, $g(\beta)$ is isomorphic to a complemented subspace of $f(\alpha') \oplus g(\beta')$. By Theorem 4.2, it follows that $g(\beta) \simeq U \oplus V$, with $U \overset{c}{\hookrightarrow} f(\alpha')$ and $V \overset{c}{\hookrightarrow} g(\beta')$. By total incomparability of $g(\beta)$ and $f(\alpha')$, U is finite dimensional. It follows that $g(\beta) \overset{c}{\hookrightarrow} U \oplus g(\beta') \simeq g(\beta')$. Symmetrically $g(\beta') \overset{c}{\hookrightarrow} g(\beta)$ and by Theorem 4.1, we deduce that $\beta =^+ \beta'$.

Similarly we get that $f(\alpha) \overset{c}{\hookrightarrow} f(\alpha') \oplus F$, where F is finite-dimensional, as well as the complemented embedding in the other direction. We may then apply Proposition 2.5 and finally get that $\alpha H_0 \alpha'$.

The claimed result is then obtained as before by circular implications and [17]. ∎

5. Final remarks, open problems and a conjecture.

Remark 5.1: Note that a Banach space not containing l_2 and without type p for some $1 \leq p < 2$ (resp. without cotype q for some $q > 2$) has at least 3 mutually non-isomorphic subspaces. Indeed: by Gowers' dichotomy theorem [15] and the fact that H.I. spaces are ergodic [30], we may assume that there exists a subspace X_1 with an unconditional basis. By the previously mentioned theorem of Komorowski and Tomczak-Jaegermann [24], some subspace X_2 of X_1 does not have an unconditional basis, but has a basis (or at least a FDD in the case when X does not have non-trivial cotype), from which it follows that it has the approximation property [25]. Finally the assumption about the type (resp. the cotype) and the results of A. Szankowski [33] imply the existence of a subspace X_3 without the approx-

imation property.

As a consequence of a study of subspaces of a Banach space with k-dimensional unconditional structure, for $k \in \mathbb{N}$, R. Anisca proved that for X non isomorphic to l_2 and with finite cotype, $l_2(X)$ has countably mutually non-isomorphic subspaces [1].

Finally it is proved in [10] that every Banach space contains a subspace which is a minimal space (that is, embeds in any of its subspaces) or contains continuum many mutually non isomorphic subspaces.

With Remark 5.1, our results and those mentioned in 1.3 of the introduction, known facts about complexity of isomorphism between subspaces of a given Banach space may be seen in Figure 2. For each equivalence relation E, we write the Banach spaces X for which we know that E is Borel reducible to isomorphism between subspaces of X.

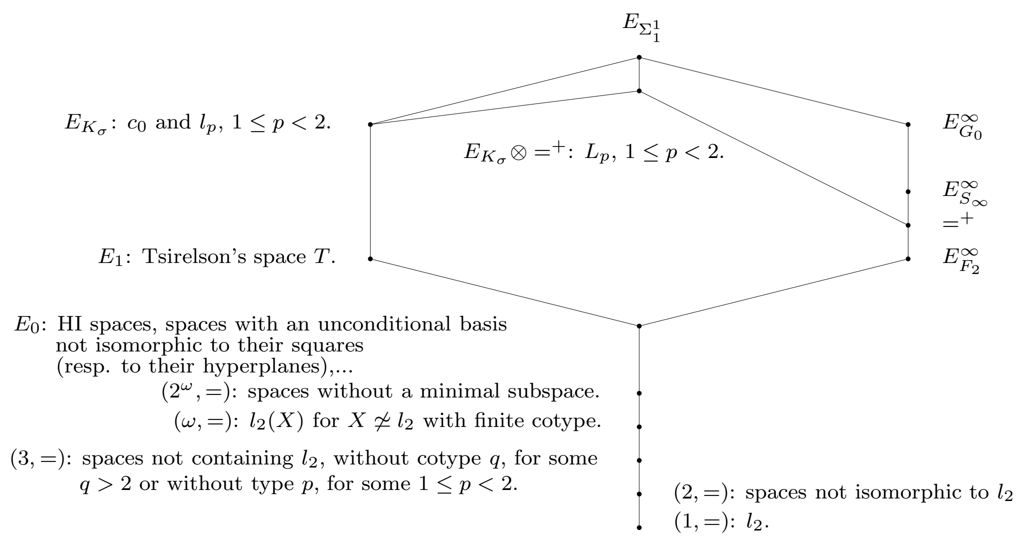

Figure 2: diagram of complexity of isomorphism between subspaces of a separable Banach space.

Remark 5.2: A problem we left open is whether we may extend our results to prove that spaces l_p are ergodic for $p > 2$. Our results about c_0 and $l_p, 1 \leq p < 2$ suggest the conjecture that l_2 is the only non-ergodic Banach space. It is also of interest to restrict the question to particular subspaces, such as block-subspaces of a given basis. As any normalized block-basis of the canonical basis of c_0 or l_p is equivalent to the original basis, these

spaces would be, as is maybe natural, of the lowest complexity possible. We conjecture the following.

Conjecture 5.3: Let X be a Banach space with an unconditional basis. Then either the relation E_0 is Borel reducible to isomorphism between block-subspaces of X, or X is isomorphic to c_0 or l_p, $1 \leq p < +\infty$.

Remark 5.4: The question of the exact complexity of isomorphism between separable Banach spaces (seen as subspaces of $C([0,1])$ with the Effros-Borel structure), or even between Banach spaces with Schauder bases (seen as subsequences of the universal basis of Pełczyński) is quite open. It could be Σ_1^1-complete, that is, $\leq_B$-maximum among analytic equivalence relations.

The limitation for our methods might come from the fact that our examples are isomorphic exactly when their canonical bases are permutatively equivalent. If the complexity of permutative equivalence of basic sequences is too low, it will be necessary to find quite different types of reductions. It could be interesting to work with general Banach lattices instead of discrete ones.

Remark 5.5: Gowers [14] solved the so-called Schroeder-Bernstein problem for Banach spaces, by proving that complemented biembeddability and isomorphism between Banach spaces need not coincide in general. We notice that for our examples, they do coincide. This is probably because many techniques known about isomorphic spaces concern, more generally, complemented subspaces. We may wonder how far these two properties are from each other from a point of view of complexity. In this direction, we show in [8] how to construct a continuum of mutually non-isomorphic subspaces which are however complemented in each other.

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Equipe d'Analyse Fonctionnelle,
Université Paris 6,
Boîte 186, 4, Place Jussieu,
75252, Paris Cedex 05,
France.
E-mail: ferenczi@ccr.jussieu.fr.

and

Department of Mathematics, IME.
University of São Paulo.
05311-970 São Paulo, SP,
Brasil.
E-mail: ferenczi@ime.usp.br, eloi@ime.usp.br